18.434 Pset 2 Due 03/4/2022

Cole Franks

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Choose 3 problems to write up. I recommend solving even the ones you don't write up. The hints are there in case you want to follow them, but you may solve the problems however you like. As usual, you may collaborate, but the writeup should be in your own words and you should list collaborators and source material.

1. What is the equality case of $\bar{d} \leq \lambda_1$, where λ_1 is the maximum eigenvalue of $A(G)$? Prove it.

2. The spectral layout of a graph $G = (V, E)$ is defined as the function $\rho : V \to \mathbb{R}^d$ minimizing $E(\rho) := \sum_{(i,j)\in E} \|\rho(i)-\rho(j)\|^2$ subject to $\sum_{v\in V} \rho(v) = 0$ and $\sum_{v\in V} \rho(v)\rho(v)^T = I_d$. Show that $E(\rho)$'s minimum value is $\lambda_2 + \cdots + \lambda_{d+1}$ where $\lambda_1 \leq \cdots \leq \lambda_n$ are the Laplacian eigenvalues of G.

3. Define the generalized hypercube graph as the Cayley graph of the d-dimensional Boolean hypercube H_d with generators $g_1, \ldots, g_k$. Show that if $g_1, \ldots, g_k$ are selected uniformly at random from H_d for $k = 10d$, then with probability tending to 1 as $d \to \infty$, all nonzero *Laplacian* eigenvalues of the generalized hypercube graph are in the range $k \pm k/\sqrt{5}$. You may make use of the **Chernoff bound:** for $X_1, \ldots, X_t$ i.i.d uniform ± 1 random variables, $\Pr[|\sum X_i| > h] \leq 2e^{-h^2/2t}$. This is interesting because the degree is only $O(\log n)$ where $n = 2^d$ is the number of vertices, but spectrally it behaves like a complete graph.

4. A column-stochastic matrix has all columns summing to 1. Show that a column-stochastic nonnegative matrix A has a nonnegative vector u summing to 1 such that $Au = u$. When is u unique?

5. Under the assumptions of the previous problem, and the further assumption that A is $n \times n$ and every entry of A is at least $(1-\alpha)/n$, prove that for any nonnegative u_0 summing to 1 we have $\|A^i u_0 - u\|_1 \leq \alpha^i \|u_0 - u\|_1$. In other words, the power method quickly finds u.