

18.434 Pset 4 Due Mon 04/18/2022

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Choose at least two problems to write up. I recommend solving even the ones you don't write up. The hints are there in case you want to follow them, but you may solve the problems however you like. As usual, you may collaborate, but the writeup should be in your own words and you should list collaborators and source material.

1. **Positive-definiteness:** For symmetric $n \times n$ matrices A and B , write that $A \succeq B$ if and only if $A - B$ is positive semidefinite, where we recall that a symmetric matrix C is positive-semidefinite if $x^T C x \geq 0$ for all x . We will show a few properties of this now (not necessarily in any particular order). Let A and B be positive-semidefinite matrices.

Note: for this course, positive-semidefinite matrices are symmetric by definition. This is just a matter of convention (but it makes sense; matrices can be written as a sum of symmetric and antisymmetric parts and definiteness only depends on the symmetric part. Moreover, complex matrices can only satisfy the definiteness condition if they are antisymmetric.)

- (a) Let $\lambda_1 \geq \dots \geq \lambda_n$ be eigenvalues of a symmetric matrix A . Then $\lambda_1 I_n \succeq A \succeq \lambda_n I_n$. **Hint:** Courant Fischer.
 - (b) Let A be positive-semidefinite. Show that there is a matrix, denoted \sqrt{A} , such that $\sqrt{A}\sqrt{A} = A$. **Hint:** use the spectral theorem.
 - (c) Suppose C is invertible and A, B symmetric. Then $CAC^T \succeq CBC^T$ if and only if $A \succeq B$.
 - (d) Suppose A and B are positive-definite. Then $A \succeq B$ if and only if $B^{-1} \succeq A^{-1}$. **Hint:** Choose a clever choice of C (possibly involving item b) and apply item c together with the fact that for any square matrices S and T , the spectrum of ST and TS are the same.
2. **Expanders:** We say a d -regular graph G is an ε -expander if every eigenvalue of the adjacency matrix $A(G)$ except λ_1 is at most εd in absolute value. Recall that G ε -approximates H if and only if the Laplacians satisfy

$$(1 + \varepsilon)L(H) \succeq L(G) \succeq (1 - \varepsilon)L(H).$$

- (a) Show that a d -regular graph G is an ε -expander if and only if G ε -approximates $(d/n)K_n$, the complete graph with edges of weight d/n . **Hint:** Courant Fischer.
3. **Harmonic functions:** A function $x : V(G) \rightarrow \mathbb{R}$ on the vertices of a graph G is said to be harmonic on $S \subset V(G)$ if for all $v \in S$ the function value at v is the average of the function values of at the neighbors of v .
- (a) Show that if G is connected then every function that is harmonic on $S = V$ is constant.
- (b) The vertices in $V(G) \setminus S$ are called *boundary* vertices and the values x takes here the boundary values. Finding a function that is harmonic on S with given boundary values is called the *Dirichlet problem*. How is this related to the spring problem from Aruzhan's talk and the voltage and random walk problems from Max, Lay, Alina and Sherri's talks? Conclude that the Dirichlet problem has a unique solution if G is connected.
- (c) Why is the Dirichlet problem not substantially different if we allow $x : V \rightarrow \mathbb{R}^n$ for some $n > 1$?
4. **Energy:** Consider a directed graph with two distinguished vertices s, t (source and sink), a.k.a. a flow network. We define a flow j_{xy} as any assignment of real numbers to a directed graph that satisfies Kirchoff's law at every point other than s, t , namely that the flow into and out of any internal vertex is zero. A unit flow has net flow out of the source equal to 1. Given an assignment of resistances R_{xy} to the edges, define the energy dissipation of a flow to be $\sum_{xy} R_{xy} j_{xy}^2$.
- Show that the unit electrical current, namely a unit flow i_{xy} which satisfies Ohm's law for some voltage v applied at source and 0 applied at sink, minimizes the energy dissipation. This is known as Thomson's principle. **Hint:** to build intuition you may want to begin with the case that all $R_{xy} = 1$. If you like physics, it might also help to interpret the resistances as spring constants.
 - Using that a unit current has energy dissipation R_{eff} (this makes sense from the formula $I^2 R$ for energy dissipation across a resistor), conclude Rayleigh's monotonicity law for effective resistance.
5. **Springs:** This is exercise 1.2.5 in Doyle and Snell. Recall that a spring network for a graph G is an assignment $x : V(G) \rightarrow \mathbb{R}$ of the vertices to real numbers and each vertex feels a force $-(x(v) - x(w))$ for every w adjacent to v . We say a spring network with a subset $F \subset V$ of fixed vertices is at equilibrium if the total force on each non-fixed vertex is zero.

The *method of relaxations* method to approximating the equilibrium of a spring network is the following iterative scheme: start with an initial guess for x , and fix an

ordering on $V \setminus F$. One at a time, replace v by the average $x(v)$ by $d(v)^{-1} \sum_{w \sim v} x(w)$ for $v \in V \setminus F$. Repeat the replacement step until you are satisfied.

- (a) Suppose the initial guess satisfies $x(v) \leq d(v)^{-1} \sum_{w \sim v} x(w)$. Show that for each v , $x(v)$ is monotone increasing throughout the process and has a limit $\tilde{x}(v)$. Show that \tilde{x} solves the spring network problem.