Matroid intersection activity

Collaborate on these with your breakout room in explain.mit.edu (or using whatever method you find convenient).

1. Let G be a bipartite graph with bipartition A, B.
   (a) Given an example showing that the set of matchings does not form the independent sets of a matroid.
   (b) Show that the set of matchings is the intersection of two matroids. **Hint:** [1]

[1] It is the intersection of two partition matroids.

2. Recall a problem from Pset 4: given an undirected graph G and an assignment p of numbers to the vertices, we'd like to direct the edges in G so that every vertex has at most $p(v)$ incoming edges.

   (a) Describe a pair of matroids whose largest common independent set has size $|E|$ if and only if G has a direction satisfying the above condition. **Hint:** [2]

[2] Again, two partition matroids will suffice.

3. Suppose G is an undirected graph and the edge set E of G has been "colored," Show that the set of *colorful spanning trees* (spanning trees whose edges are all different colors) is the set of common bases of two matroids. **Hint:** [3]

[3] This time you can use a graphic matroid and a partition matroid.

4. For a directed graph D and a "root" vertex $r \in V(D)$ such that r has no incoming edges, define an *arborescence* to be a spanning tree of D directed away from r.[4]

   (a) Let G be the underlying undirected graph of D obtained by forgetting the directions of the edges.[5] Show that any subgraph of D which corresponds to a spanning tree in G and has at most one edge entering each vertex is an arborescence.

   (b) Show that the set of arborescences of D, r is the set of common bases of two matroids. **Hint:** [6]

[4] Here spanning tree just means that it's a spanning tree in the underlying undirected graph G.
[5] G may have multi-edges if both directions (u, v) and (v, u) of an edge were present in $E(D)$.
[6] Again you can use the intersection of a graphic matroid and a partition matroid.

5. Consider an undirected graph G. We'd like to decide if G is the union of two edge-disjoint spanning trees. Given a matroid $M = (E, I)$, define its *dual matroid* M^* to be (E, I^*) where I^* is the set of subsets of E whose complements contain a base of M.

   (a) Describe a pair of matroids whose largest common independent set has size $|V| - 1$ if and only if G has two edge-disjoint spanning trees. You may use that the dual matroid is indeed a matroid.

   (b) **Bonus:** prove that the dual matroid is a matroid.