18.453 Quiz

Instructions. This is a **practice, take-home** quiz. This is meant to be done in **2** hours with access to notes and course material, but no access to collaborators. For best practice I suggest trying to complete it under these conditions. Afterwards please tell me if 2 hours felt like enough.

1. Consider a bipartite graph $G = (V, E)$ in which every vertex has degree k (a so-called k-regular bipartite graph). Prove that such a graph always has a perfect matching in two different ways:

   (a) by using König's theorem,

   (b) by using the LP formulation of the min-weight perfect matching problem.

2. Suppose $G = (V, E)$ is a 2-edge-connected graph (that is, G remains connected if you delete any single edge) with at least one perfect matching, and suppose that G has a special edge e such that the graph obtained by removing e from G has no perfect matching. Show that there is necessarily a nonempty set $S \subseteq V$ with the following properties:

   - the number of odd components of $G \setminus S$ is exactly $|S|$,
   - $G \setminus S$ has at least one even component.

3. Show that for any point x_0 in an unbounded polyhedron $P \subset \mathbb{R}^n$, P contains a *ray* from x_0, a set of the form $\{x_0 + \alpha y : \alpha \geq 0\}$ for some $y \in \mathbb{R}^n$. A suggested approach:

   - Show it is enough to prove this when x_0 is 0.
   - As P unbounded, there is some c such that $\max\{c^T x : x \in P\} = \infty$. Apply linear programming duality for the linear program $\max\{c^T x : x \in P\}$ to show something about the feasibility/infeasibility of the dual.
   - Apply Farkas' lemma to the infeasibility/feasibility of the dual in order to obtain the direction of the ray.

An extra problem: This one shouldn't count as part of your 2 hours, but a problem like it could appear on the exam.

4. Let G be a bipartite graph with bipartition A, B and edge set E. A *fractional vertex cover* is a pair of assignments of numbers $(x_a \in \mathbb{R} : a \in A)$ and $(y_b \in \mathbb{R} : b \in B)$ to the vertices such that

$$\begin{aligned} x_a + y_b &\geq 1 \quad \forall ab \in E \\ x_a &\geq 0 \quad \forall a \in A \\ y_a &\geq 0 \quad \forall b \in B \end{aligned}$$

The *fractional vertex cover number* is

$$\tau(G) := \min \left\{ \sum_{a \in A} x_a + \sum_{a \in B} y_a : (x, y) \text{ is a fractional vertex cover of } G \right\}.$$

Show that the fractional vertex cover number is the same as the vertex cover number, i.e. the size of a minimum vertex cover. **Hint:** [1]

[1] Use total modularity, and that M^T is totally unimodular if and only if M^T is.