The Δ^2 conjecture holds for graphs of small order. *

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Abstract

An $L(2,1)$-labeling of a simple graph G is a function $f : V(G) \to \mathbb{Z}$ such that if $xy \in E(G)$, then $|f(x) - f(y)| \geq 2$ and if the distance between x and y is two then $|f(x) - f(y)| \geq 1$. $L(2,1)$-labelings are motivated by radio channel assignment problems. $\lambda_{2,1}(G)$ is the smallest integer such that there exists an $L(2,1)$-labeling of G using the integers $\{0, ..., \lambda_{2,1}(G)\}$. We prove that $\lambda_{2,1}(G) \leq \Delta^2$, where $\Delta = \Delta(G)$, if the order of G is no greater than $(\lfloor \Delta/2 \rfloor + 1)(\Delta^2 - \Delta + 1) - 1$. This shows that for graphs no larger than the given order, the 1992 "Δ^2 Conjecture" of Griggs and Yeh holds. In fact, we prove more generally that if $L \geq \Delta^2 + 1$, $\Delta \geq 1$, and

$$|V(G)| \leq (L - \Delta)\left(\left\lfloor \frac{L-1}{2\Delta} \right\rfloor + 1\right) - 1,$$

then $\lambda_{2,1}(G) \leq L - 1$. In addition, we exhibit an infinite family of graphs with $\lambda_{2,1}(G) = \Delta^2 - \Delta + 1$.

1 Introduction

The *channel assignment problem* is the determination of assignments of channels (integers) to stations such that those stations close enough to interfere receive distant enough channels. Hale [8] formulated the problem in terms of T-colorings, which are integer colorings in which adjacent vertices' colors cannot differ by a member of a set of integers T with $\{0\} \subset T$. Roberts [15] proposed a generalization in which closer transmitters would be required to have channels that differed by more than those of the slightly more distant transmitters, adding a condition for non-adjacent vertices as well. The $L(2,1)$-labeling problem was first studied by Griggs and Yeh in 1992 in response to Roberts' proposal. An $L(2,1)$-labeling of a graph G is an integer labeling of G in which two vertices at distance one from each other must have labels differing by at least 2, and those at distance two must differ by at least 1. $\lambda_{2,1}(G)$ is the smallest number such that there exists an $L(2,1)$-labeling of G with the difference $\lambda_{2,1}(G)$ between the highest and lowest label. $\lambda_{2,1}(G)$ is sometimes written

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$\lambda_{2,1}$ if there is no possibility for confusion. The $L(2,1)$-labeling problem has been studied extensively, with the central goal of finding bounds on $\lambda_{2,1}$. Griggs and Yeh bounded the $\lambda_{2,1}$ number for cycles, paths, trees, and the n-cube in their original 1992 paper[6]. They also proved the bound $\lambda_{2,1} \leq \Delta(G)^2 + 2\Delta(G)$, where $\Delta(G)$ is the maximum degree over the set of degrees of vertices in $V(G)$. In this paper we will write Δ when the meaning is clear from context. Chang and Kuo [2] improved the bound to $\Delta^2 + \Delta$, and Gonçalves [5] reduced the bound to $\Delta^2 + \Delta - 2$ by modifying the algorithm of Chang and Kuo. Bounds on the $\lambda_{2,1}$ number have been found for many subclasses of graphs, such as Sakai's bound of $(\Delta + 3)^2/4$ for *chordal* graphs - graphs containing no induced cycle of length four [16]. All examples tested have corroborated the conjecture Griggs and Yeh made in their 1992 paper:

Δ^2 **Conjecture.** If $\Delta(G) \geq 2$, then $\lambda_{2,1} \leq \Delta^2$.

However, the conjecture remains unproven, and it is difficult to test the bound for graphs of any significant size. The largest step towards the proof of the conjecture was made by Havet, Reed, and Sereni, who proved that the the conjecture holds for all graphs with Δ larger than some Δ_0, but $\Delta_0 \approx 10^{69}$ [9]. Consequently, $\lambda_{2,1}(G) \leq \Delta^2 + C$ for some absolute constant C. The upper bound set by the conjecture, if proven, would be tight - the Moore Graphs are known to satisfy $\lambda_{2,1} = \Delta^2$ [6].

2 Preliminaries

The proof of Theorem 1 involves a classic result of Pósa about the existence of Hamilton cycles and paths in graphs of high degree [13]. In this respect, our argument has a similar flavor to the proof in [6] that $\lambda_{2,1} \leq \Delta^2$ for graphs of order less than $\Delta^2 + 1$. In addition, we will use the powerful result of Szemerédi and Hajnal on equitable colorings [7].

Theorem (Pósa). Let G have $n \geq 3$ vertices. If for every k, $1 \leq k \leq (n-1)/2$, $|\{v : d(v) \leq k\}| < k$, then G is Hamiltonian [13].

Corollary 1. Let G have $n \geq 2$ vertices. If for every k, $0 \leq k \leq (n-2)/2$, $|\{v : d(v) \leq k\}| \leq k$, then G has a Hamilton path.

Proof. The corollary follows easily by adding a dominating vertex to G and observing that by Pósa's Theorem the new graph is Hamiltonian. $\square$

Theorem (Szemerédi, Hajnal). If $\Delta(G) \leq r$, then G can be equitably colored with $r+1$ colors; that is, the sizes of the color classes differ by at most one [7], [12], [11].

3 Main Result

The following lemma is the key ingredient in the proof of the main result. The lemma requires a concept which we will call the square color graph. Let G be a graph. Let $C_0, ..C_{l-1}$ be the color classes of a proper coloring C of G^2 with l colors. G^2 is the graph with $V(G^2) = V(G)$ and $E(G^2) = \{xy|d(x,y) \leq 2\}$. The *square color graph* of C, denoted $\mathcal{G}$, is the graph with

$$V(\mathcal{G}) = \{C_0, ..., C_{l-1}\}, \text{ and } E(\mathcal{G}) = \{C_iC_j \mid G[C_i \cup C_j] \text{ contains an edge of } G\}.$$

Here $G[C_i \cup C_j]$ denotes the induced subgraph formed by the vertices in $C_i \cup C_j$.

Lemma 1. Let G be a graph, and let C be a proper coloring of G^2 with l colors. If the complement $\mathcal{G}^c$ of the square color graph of C has a Hamilton path, then $\lambda_{2,1}(G) \leq l - 1$.

Proof. By assumption, $\mathcal{G}^c$ has a Hamiltonian path $P = \{p_0, p_1, ..., p_{l-1}\}$. Recall that the vertices of P are color classes partitioning G. Let $f : V(G) \to \mathbb{Z}$ be defined as $f : v \mapsto i$ where i is the unique index such that $v \in p_i$. We now check that f is an $L(2,1)$-labeling of G. If $d(x,y) = 2$, then x and y are given two different labels because C is a coloring of G^2. If $d(x,y) = 1$, then x and y are in two distinct color classes p_i and p_j such that $p_ip_j \in E(\mathcal{G})$. Then $p_ip_j \notin E(\mathcal{G}^c)$, so $i \neq j \pm 1$ because otherwise $p_ip_j \in E(P)$. Therefore $|f(x) - f(y)| \geq 2$, and f is an $L(2,1)$-labeling for G. $\square$

Theorem 1. Let G be a graph with $\Delta = \Delta(G) \geq 1$, and let L be an integer with $L \geq \Delta^2 + 1$. Then $\lambda_{2,1}(G) \leq L - 1$ if

$$|V(G)| \leq (L - \Delta)\left(\left\lfloor \frac{L-1}{2\Delta} \right\rfloor + 1\right) - 1.$$

Before the proof of Theorem 1, we will discuss two corollaries that have implications for the Δ^2 Conjecture.

Corollary 2. Let G be a graph of with $\Delta = \Delta(G) \geq 1$. Then $\lambda_{2,1}(G) \leq \Delta^2$ if

$$|V(G)| \leq \left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1\right)\left(\Delta^2 - \Delta + 1\right) - 1.$$

Proof. Using Theorem 1 with $L = \Delta^2 + 1$ gives the desired result. $\square$

Corollary 2 significantly expands the known orders of graphs that satisfy the Δ^2 Conjecture; it does so more dramatically as $\Delta(G)$ increases. For $\Delta(G) = 3$, $|V(G)| \leq 13$ suffices as opposed to the previously known $|V(G)| \leq 10$ [6]. For $\Delta(G) = 4$, we have $|V(G)| \leq 38$ as opposed to $|V(G)| \leq 17$ [6]. If G is the Hoffman-Singleton graph, then $\Delta(G) = 7$, $|V(G)| = 50 = \Delta^2 + 1$, and in fact $\lambda_{2,1}(G) = 49 = \Delta^2$ [6]. It might seem productive to look among minor variations of the Hoffman-Singleton graph for counterexamples to the

Δ^2 Conjecture, but Corollary 2 suggests otherwise - the conjecture holds if $\Delta(G) = 7$ and $|V(G)| \leq 169$. The bounds on $|V(G)|$ established in Corollary 2 grow quickly with Δ, as they are cubic in Δ rather than quadratic as in [6].

For some $|V(G)|$, we can also use Theorem 1 to find stronger upper bounds on $\lambda_{2,1}(G)$ than the best known bound of Gonçalves [5]. The bound on $|V(G)|$ in the following corollary is larger than the bound in Theorem 1.

Corollary 3. Let G be a graph with $\Delta = \Delta(G) \geq 3$. Then $\lambda_{2,1}(G) < \Delta^2 + \Delta - 2$ if

$$|V(G)| \leq \left(\left\lfloor \frac{\Delta}{2} \right\rfloor + 1 \right) (\Delta^2 - 2) - 1.$$

Proof. Apply Theorem 1 with $L = \Delta^2 + \Delta - 2$. This gives

$$|V(G)| \leq \left(\left\lfloor \frac{\Delta}{2} + \frac{1}{2} - \frac{3}{2\Delta} \right\rfloor + 1 \right) (\Delta^2 - 2) - 1.$$

Since we have assumed $\Delta \geq 3$, we have $0 \leq 1/2 - 3/(2\Delta) < 1/2$, so

$$\left\lfloor \frac{\Delta}{2} + \frac{1}{2} - \frac{3}{2\Delta} \right\rfloor = \left\lfloor \frac{\Delta}{2} \right\rfloor.$$

□

We now proceed to the proof of Theorem 1.

Proof. Let L be as in Theorem 1. We will show that for any integers $q \geq 0$, $0 \leq r \leq L - 1$ with

$$Lq + r \leq M = (L - \Delta) \left(\left\lfloor \frac{L-1}{2\Delta} \right\rfloor + 1 \right) - 1,$$

if $|V(G)| = Lq + r$ and $\Delta(G) = \Delta$ then G has an $L(2,1)$-labeling with span at most $L-1$. This is sufficient to prove Theorem 1, as for any integer n there exist unique integers $q \geq 0$ and $r \in \{0, ..., L-1\}$ with $Lq + r = n$. Suppose $|V(G)| = Lq + r$. Recall that $L \geq \Delta^2 + 1 \geq \Delta(G^2) + 1$. By the Szemerédi-Hajnal theorem, G^2 has an equitable coloring C with L color classes. For convenience we will use all L color classes even if several are empty. This means $L - r$ classes have q vertices and r classes have $q + 1$ vertices. Our goal is to prove that the complement of the square color graph of C, or $\mathcal{G}^c$, has a Hamiltonian path. Note that $d_{\mathcal{G}}(V) \leq \Delta|V|$ for all $V \in V(\mathcal{G})$. Write the degree of V in $\mathcal{G}^c$ as $d_c(V)$.

If $q \leq \lfloor (L-1)/2\Delta \rfloor - 1$, then

$$\Delta(q+1) \leq \Delta \left\lfloor \frac{L-1}{2\Delta} \right\rfloor \leq \left\lfloor \frac{L-1}{2} \right\rfloor$$

so that $\delta(\mathcal{G}^c) \geq L-1-\lfloor (L-1)/2 \rfloor \geq (L-1)/2$, and the conditions of Corollary 1 are satisfied. Therefore $\mathcal{G}^c$ has a Hamiltonian path.

Otherwise, $q = \lfloor (L-1)/2\Delta \rfloor$ and

$$r \leq L-1-\Delta\left(\left\lfloor \frac{L-1}{2\Delta} \right\rfloor + 1\right) \leq L-1,$$

because otherwise $Lq + r > M$.

Now suppose k is an integer with $0 \leq k \leq (L-2)/2$ as in Corollary 1. If $d_c(V) \leq k$, then

$$\frac{L-2}{2} \geq L-1-d_{\mathcal{G}}(V) \geq L-1-\Delta|V|,$$

so that $|V| \geq (1/\Delta)(L-1-(L-2)/2) = (L-1)/2\Delta + 1/2\Delta > q$. Therefore $|V| = q+1$, so we know there are at most r vertices with $d_c(V) \leq k$. For any such vertex V,

$$d_c(V) \geq L-1-(q+1)\Delta = L-1-\Delta\left(\left\lfloor \frac{L-1}{2\Delta} \right\rfloor + 1\right) \geq r \geq 0.$$

Now the conditions of Corollary 1 are satisfied, so $\mathcal{G}^c$ still has a Hamiltonian path. From Lemma 1, $\mathcal{G}^c$ having a Hamiltonian path implies that $\lambda_{2,1}G \leq L-1$. As

$$Lq + L - 1 - \Delta\left(\left\lfloor \frac{L-1}{2\Delta} \right\rfloor + 1\right) = (L-\Delta)\left(\left\lfloor \frac{L-1}{2\Delta} \right\rfloor + 1\right) - 1 = M,$$

this argument works for any $|V(G)| \leq M$. □

Corollary 4. Let G be a graph of order n with $\Delta = \Delta(G) \geq 1$, and let L be an integer with $L \geq \Delta^2 + 1$. If

$$n \leq (L-\Delta)\left(\left\lfloor \frac{L-1}{2\Delta} \right\rfloor + 1\right) - 1,$$

then there is an $L(2,1)$-labeling of G with a span at most $L-1$ that is equitable. If $n \geq L$, the labeling is no-hole.

Proof. The proof follows immediately from the proof of Theorem 1. □

The next corollary concerns algorithms involved in finding these labelings. In general, determining if $\lambda_{2,1}(G) \leq k$ for positive integers $k \geq 4$ is NP-complete [4].

Corollary 5. Let G be a graph of order n with $\Delta = \Delta(G) \geq 1$, and $L \geq \Delta^2 + 1$. There is an algorithm with polynomial running time in n to compute an $L(2,1)$-labeling of G with span at most $L - 1$ for all n and L such that

$$n \leq (L - \Delta)\left(\left\lfloor \frac{L-1}{2\Delta} \right\rfloor + 1\right) - 1.$$

Proof. If $L \geq 2n + 1$, the appropriate labeling can be obtained by labeling the vertices $0, 2, ..., 2n$ in any order [6]. This can clearly be done in polynomial time. Otherwise, in [12] there is shown to be an algorithm polynomial in n to equitably color G^2 with L colors. Degree sequences satisfying the conditions of Pósa's Theorem also satisfy those of Chvátal's Theorem [1], and the paper's authors exhibit an algorithm polynomial in p to find Hamilton cycles in graphs of order p which satisfy the conditions of Chvátal's Theorem. From the proof of Lemma 1 and of Corollary 1, we see that to find the labeling it is enough find a Hamilton cycle in a certain graph, namely $\mathcal{G}^c$ with a dominating vertex added, of order $L + 1 \leq 2n + 2$ that satisfies the conditions of Pósa's Theorem. From [1], we can do this with an algorithm polynomial in $L + 1$, which must also be polynomial in n. These two algorithms in succession yield the desired algorithm. □

4 Comments on Diameter Two Graphs

It was previously known that diameter-two graphs satisfy the Δ^2 conjecture, and for other than a few exceptional graphs, $\Delta^2 - 1$ suffices to label diameter two graphs [6]. In this section we knock this bound down by one, showing that $\Delta^2 - 2$ suffices to label all but a finite handful of diameter two graphs.

Theorem 2 (Griggs, Yeh [6])**.** The Δ^2 Conjecture holds for diameter two graphs. In addition, $\lambda_{2,1} \leq \Delta^2 - 1$ for diameter two graphs with $\Delta \geq 2$ except for C_3, C_4 and the Moore Graphs. For these exceptional graphs, $\lambda_{2,1} = \Delta^2$.

The proof of these facts rely on Brooks' Theorem and several results from Griggs and Yeh:

Theorem 3 (Brooks [14])**.** $\chi(G) \leq \Delta + 1$, and $\chi(G) \leq \Delta$ unless G is an odd cycle or a complete graph.

Lemma 2 (Griggs, Yeh [6])**.** $\lambda_{2,1}(G) \leq |V(G)| + \chi(G) - 2$.

Lemma 3 (Griggs, Yeh [6])**.** There exists an injective $L(2,1)$-labeling of a graph G with span $|V(G)| - 1$ if and only if the complement of G has a Hamilton path.

Theorem 4 (Griggs, Yeh [6])**.** Let C_n be a cycle on n vertices. Then $\lambda_{2,1}(C_n) = 4$.

We now proceed to prove Theorem 2.

Proof. If $\Delta = 2$, one can verify the theorem readily using Theorem 4. Suppose $\Delta \geq 3$. We now split into cases.

In the first case, suppose $\Delta \geq (|V(G)|)/2$. Lemma 2 implies $\lambda_{2,1}(G) \leq 2\Delta + \chi(G) - 2$. If G is a complete graph, then clearly $\lambda_{2,1}(G) = 2\Delta(G)$. G is not an odd cycle, as $\Delta \geq 3$. Otherwise, $2\Delta + \chi(G) - 2 \leq 3\Delta - 2$ by Brooks' Theorem. Note that in both cases, $\Delta(G) \geq 3$ implies that $\lambda_{2,1}(G) \leq \Delta^2 - 2$.

In the second case, suppose $\Delta \leq (|V(G)| - 1)/2$. Then $\delta(G^c) \geq (|V(G)| - 1)/2$. Also, we have assumed G has $\Delta \geq 3$, so $|V(G)| \geq 7$. By Corollary 1, G^c has a Hamilton path. By Lemma 3, there is an $L(2,1)$-labeling of G with span $|V(G)| - 1$. As the Moore graphs are the only diameter two graphs with $|V(G)| = \Delta^2 + 1$, Theorem 2 holds. □

In fact, we can do better by the following result:

Theorem 5 (Erdős, Fajtlowicz, Hoffman [3])**.** Except C_4, there is no diameter two graph of order Δ^2.

This and the proof of Theorem 2 imply the following theorem.

Theorem 6. With the exception of C_3, C_4, C_5 and the Moore Graphs, any diameter two graph with $\Delta(G) \geq 2$ has $\lambda_{2,1}(G) \leq \Delta^2 - 2$.

We also have some comments on a special family of diameter two graphs that have large $\lambda_{2,1}$ number. In order to do this, we must define the points of the *Galois Plane*, denoted $PG_2(n)$. Let F be a finite field of order n. Let $P = F^3 \setminus \{(0,0,0)\}$. Define an equivalence relation $\equiv$ on P by $(x_1, x_2, x_3) \equiv (y_1, y_2, y_3) \iff (x_1, x_2, x_3) = (cy_1, cy_2, cy_3)$ for some $c \in F$. The *points* of $PG_2(n)$ are the equivalence classes.

Definition 1. The *polarity graph of* $PG_2(n)$, denoted H, is the graph with the points of $PG_2(n)$ as vertices and with two vertices (x_1, x_2, x_3) and (y_1, y_2, y_3) adjacent if and only if $y_1x_1 + y_2x_2 + x_3x_3 = 0$.

By the properties of $PG_2(n)$, we know that the diameter of H is two, $\Delta(H) = n + 1$ and its order is $n^2 + n + 1 = \Delta^2 - \Delta + 1$ [10]. This implies that $\lambda_{2,1}(H) \geq \Delta^2 - \Delta$. In fact, Yeh showed that $\lambda_{2,1}(H) = \Delta^2 - \Delta$ [6]. This is an infinite family of graphs, as finite fields exist for $n = p^k$ with p prime.

However, we can improve this by one. This construction follows Erdős, Fajtlowicz and Hoffman [3]. A vertex (x, y, z) in H has degree n if and only if the norm $x^2 + y^2 + z^2 = 0$. Suppose F has characteristic 2 and the order of F is n. If (a, b, c) is in H then it is adjacent to the point $(b + c, a + c, a + b)$ which has norm 0 and is also in H. In other words, every vertex in H is adjacent to a vertex of degree n. We proceed to find the number of points of degree n in H. Since the characteristic of $F = 2$, $f(x) = x^2$ is injective and hence surjective on F. This means we can choose x^2 and y^2 freely as as long as one of them is nonzero, and then z^2 is determined. We must also eliminate proportional pairs, so in total this leaves $(n^2-1)/(n-1) = n+1$ vertices of degree n.

Now we can make an $(n+1)$-regular, diameter two graph $\tilde{H}(n)$ by adding a vertex that is adjacent to all vertices of degree n. This graph is of order $n^2+n+2 = \Delta^2 - \Delta + 2$.

Theorem 7. The graph $\tilde{H}(n)$ has $\lambda_{2,1}(\tilde{H}) = \Delta^2 - \Delta + 1$.

Proof. Because $\tilde{H}$ is diameter two, $\lambda_{2,1}(\tilde{H}) \geq \Delta^2 - \Delta + 1$. As $\Delta \geq 3$, $\Delta \leq (\Delta^2 - \Delta + 1)/2 = (|V(H)| - 1)/2$. By the proof of Theorem 2, $\lambda_{2,1}(\tilde{H}) \leq |V(G)| - 1 = \Delta^2 - \Delta + 1$. □

$\tilde{H}(n)$ exists for all $n = 2^k$, so this is an infinite family of graphs.

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