18.434 Pset 1 Due 02/21/2020

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February 19, 2020

Choose 3 problems to write up. I recommend solving even the ones you don't write up. The hints are there in case you want to follow them, but you may solve the problems however you like. As usual, you may collaborate, but the writeup should be in your own words and you should list collaborators and source material.

1. On 2/06 (Lecture 2) we calculated the spectrum of the path P_n, but we did not calculate the *Laplace spectrum* of the path. That is, the spectrum of $L(P_n)$, the Laplacian of the path. That's the purpose of this exercise. Feel free to try with or without the below plan, which is an expanded version of the argument in BH 1.4.4 and also in Spielman's lecture on paths, rings, and Cayley graphs.

   Hint: The idea here will be very similar to embedding $L(P_n)$ into a large matrix, except instead of a submatrix we will be expressing $L(P_n)$ as $\pi M \pi^T$ where π is a partial isometry (a matrix π such that $\pi\pi^T = I_n$). The matrices $\pi M \pi^T$ generalize the submatrices of M because, for instance, the matrix $P_r : \mathbb{R}^n \to \mathbb{R}^r$ sending a vector to its first r coordinates is a partial isometry and $P_r M P_r^T$ is the $r \times r$ submatrix in the top left corner of M.

   (a) Suppose $\pi : \mathbb{R}^n \to \mathbb{R}^r$ is a partial isometry and that $\pi^T v \in \mathbb{R}^n$ is an eigenvector of M for some $v \in \mathbb{R}^r$. Show that v is an eigenvector of the $r \times r$ matrix $\pi M \pi^T$ with the same eigenvalue.

   (b) Show that $L(P_n)$ is $\pi_n L(C_{2n}) \pi_n^T$ where $\pi_n : \mathbb{R}^{2n} \to \mathbb{R}^n$ is the partial isometry

   $$\pi_n := \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} I_n & H_n \end{array} \right],$$

   whee H_n is an identity matrix rotated 90 degrees. e.g.

   $$\pi_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

   (c) What vectors $w \in \mathbb{R}^{2n}$ are of the form $\pi_n^T v$ for some $v \in \mathbb{R}^n$? Find n eigenvectors of $L(C_{2n})$ with this property.

(d) Find an orthogonal eigenbasis for $L(P_n)$ and find its spectrum.

2. Show the formula

$$x^T L(G)x = \sum_{i,j \in E} (x_i - x_j)^2.$$

An object from Collin's talk might help.

3. Show that if v is in the positive eigenspace of a symmetric matrix A, i.e. the sum of the eigenspaces corresponding to positive eigenvalues, then $x^T Ax > 0$.

4. Show that the average degree $\bar{d}$ of G is a lower bound for the largest eigenvalue λ_1 of $A(G)$. **Hint:** use Courant-Fischer with an intelligently chosen test vector.

5. Say a nonnegative matrix T is *primitive* if $T^k > 0$ for some integer $k \geq 0$. Show that all the other eigenvalues are strictly smaller in absolute value than the absolute value of the largest positive one.