# 18.434 Pset 2 Due 03/6/2020

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**Choose 3 problems to write up.** I recommend solving even the ones you don't write up. The hints are there in case you want to follow them, but you may solve the problems however you like. As usual, you may collaborate, but the writeup should be in your own words and you should list collaborators and source material.

1. Show that the algebraic connectivity of a graph is superadditive under disjoint unions, i.e. for $G, H$ edge-disjoint on the same vertex set we have $\mu_2(G+H) \geq \mu_2(G)+\mu_2(H)$ where $G + H$ is the graph whose edge set is the union of those of $G, H$.

   **Remark.** Actually, Laplacians make sense for edge-weighted graphs, so if $G$ and $H$ are weighted graphs and I define $G + H$ to have each edge weighted by the sum of the weight on the edge in $G$ and the weight on the edge in $H$ (thinking of non-edges as zero weight edges) then the algebraic connectivity (defined spectrally) is still super-additive, even if $G$ and $H$ aren't disjoint.

2. Using the fact that the spectra of principal submatrices interlace, prove the statement used by Max:

   **Theorem.** If $A$ is a symmetric $n \times n$ matrix and $x_1, \dots, x_m \in \mathbb{R}^n$ are orthogonal, then the spectrum of the $m \times m$ matrix $C_{i,j} := \frac{1}{\|x_i\|^2} x_i^T A x_j$ interlaces that of $A$.

   **Hint:** Make a change of basis and then apply the submatrix version of interlacing to show $B_{i,j} := \frac{1}{\|x_i\|\|x_j\|} x_i^T A x_j$ interlaces. Then show $C, B$ are similar.

3. Prove the upper inequality in Theorem 3.7.4 in the book, namely that the Lovasz $\theta$ number is at most the chromatic number of the complement. **Hint:** The proof is in the book, but to successfully answer this question you should explain what's going on in that proof in your own words.

For fun I also recommend reading the proof of Proposition 3.7.5.

4. Define the inner product between two characters[1] $\phi, \psi$ of a finite group $G$ as

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)},$$

where if $z$ is a complex number then $\overline{z}$ denotes its complex conjugate. Show

- the characters of a group (thought of as vectors indexed by group elements) are eigenvectors of any Cayley graph of $G$, and that
- the set of characters is orthonormal.

**Hint for second part:** First show that the characters of a finite group take values in the unit circle in the complex plane. Then $\bar{\psi(g)} = \psi(g)^{-1}$. Use this to show that the inner product between two distinct characters is equal to itself times a number that isn't 1, so it must be zero.

5.

[1] I mean multiplicative character, as defined in Agustin's talk as a homomorphism from $G$ to the complex numbers with multiplication, not character in the sense of representation theory. For Abelian groups these two notions coincide, and the characters form an orthonormal basis for the vector space of complex functions on the group (and hence an orthonormal eigenbasis for the Cayley graph.)