18.434 Pset 3 Due Wed 03/30/2020

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Choose at least two problems to write up. I recommend solving even the ones you don't write up. The hints are there in case you want to follow them, but you may solve the problems however you like. As usual, you may collaborate, but the writeup should be in your own words and you should list collaborators and source material.

1. Let G be a d-regular graph. Prove that for every vertex v in G, the number of walks in G from v to v of length k is at least the number of closed walks of length k from the root to itself in the infinite d-regular tree.

2. Suppose G is a d-regular graph and let $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$. Let S and T be disjoint subsets of the vertices of G. Show that the total number $W(S,T)$ of walks from S to T of length k (the number of k-step walks starting at some vertex in S and ending at some vertex in T) satisfies

$$\left| W(S,T) - \frac{d^k}{n}|S||T| \right| \leq \lambda^k \sqrt{|S||T|}.$$

   Hint: Express $W(S,T) = \chi_S^T A^k \chi_T$ and proceed as in the proof of the expander mixing lemma.

3. Let A be a matrix with eigenvalues $\lambda_1 = 1, \lambda_2, \ldots, \lambda_n$ and suppose that P is an invertible matrix such that PAP^{-1} is a symmetric matrix. Let $\lambda := \max_{i \neq 1} |\lambda_i|$ and v_1 be an eigenvector of A with eigenvalue 1. Show that if $\lambda < 1$ then for all $v \in \mathbb{R}^n$ the sequence $A^k v$ converges to

$$\left(\frac{\langle P v_1, P v \rangle}{\|P v_1\|^2} \right) v_1.$$

   Discuss the relevance of this to a lazy random walk on an undirected graph.