Analytic algorithms for the moment polytope

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Based on joint work with

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Outline

1. Moment polytopes by example
2. Algorithms for the general problem

Moment polytopes

Motivating question

Horn's problem:

Are $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3 \in \mathbb{R}^n$ the spectra of three $n \times n$ matrices H_1, H_2, H_3 such that

$$H_1 + H_2 = H_3?$$

If so, can one find the matrices efficiently?

Horn set

Let $\mathcal{V} = \mathbb{P}(\mathrm{Mat}(n)^2)$, define

$$\mu : \mathcal{V} \to \mathrm{Herm}(n)^3$$

by

$$\mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, \quad A_2 A_2^\dagger, \quad A_1^\dagger A_1 + A_2^\dagger A_2)}{\|A_1\|^2 + \|A_2\|^2}.$$

Note $\mathrm{eigs}(AA^\dagger) = \mathrm{eigs}(A^\dagger A)$, so

$$\mathrm{eigs}(A_1 A_1^\dagger), \quad \mathrm{eigs}(A_2 A_2^\dagger), \quad \mathrm{eigs}(A_1^\dagger A_1 + A_2^\dagger A_2)$$

is a “yes” instance to Horn’s problem (in fact, all such instances take this form).

Moment polytopes

- $G = \mathrm{GL}(n)$
- $\pi : G \to \mathbb{C}^m$ a representation of G where $U(n)$ acts unitarily
- $\mathcal{V} \subset \mathbb{P}(\mathbb{C}^m)$ a projective variety fixed by G,

Moment map is the map $\mu : \mathcal{V} \to n \times n$ Hermitians $=: \mathrm{Herm}(n)$ given by

$$\boxed{\mu : v \mapsto \nabla_{H \in \mathrm{Herm}(n)} \log \|e^H \cdot v\|}$$

$i\mu$ is a moment map for $U(n)$ in the physical sense! In particular:

Theorem (Kirwan)

Image of

$$\mathcal{V} \xrightarrow{\quad \mu \quad} \mathrm{Herm}(n) \xrightarrow{\text{take eigs.}} \mathbb{R}^n$$

is a convex polytope in $\mathbb{R}^n$ known as moment polytope, denoted $\Delta(\mathcal{V})$

Horn polytope

- $\mathcal{V} = \mathbb{P}(\mathrm{Mat}(n)^2)$
- $G = \mathrm{GL}(n)^3$
- π given by

$$(g_1, g_2, g_3) \cdot (A_1, A_2) = (g_1 A_1 g_3^\dagger, g_2 A_2 g_3^\dagger).$$

- $\mu : \mathcal{V} \to \mathrm{Herm}(n)^3$ given by

$$\mu : [A_1, A_2] \mapsto \frac{(A_1 A_1^\dagger, \quad A_2 A_2^\dagger, \quad A_1^\dagger A_1 + A_2^\dagger A_2)}{\|A_1\|^2 + \|A_2\|^2}.$$

Thus, image of

$$\mathcal{V} \xrightarrow{\quad \mu \quad} \mathrm{Herm}(n)^3 \xrightarrow{\text{take eigs.}} (\mathbb{R}^n)^3$$

is the* solution set of the Horn problem!

Link to algebra

[CF: Missing!]

Algorithmic tasks

Input $(\mathcal{V}, \pi, \lambda)$

- Projective variety $\mathcal{V}$ as arithmetic circuit parametrizing it
- Representation π as its list of irreducible subrepresentations as elements of $\mathbb{Z}^n$
- Target $\lambda \in \mathbb{Q}^n$

1. **membership:** determine whether $\boldsymbol{\lambda}$ in $\Delta(\mathcal{V})$.
2. ε**-search:** given $\boldsymbol{\lambda} \in \mathbb{R}^n$, either find an element $v \in \boldsymbol{\lambda}$ such that
   - $\|\mu(v) - \mathrm{diag}(\boldsymbol{\lambda})\| < \varepsilon$, OR
   - correctly declare $\boldsymbol{\lambda} \notin \Delta(\mathcal{V})$.

   i.e. find an approximate preimage under μ!

$1/exp$(poly)-search suffices for membership!

Algorithm for ε-search for Horn polytope (F18)

Input: $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3) \in (\mathbb{R}^n)^3$ and $\varepsilon > 0$.

1. Choose A_1, A_2 at random. Define

$$\mu_1 = A_1 A_1^\dagger, \quad \mu_2 = A_2 A_2^\dagger, \quad \mu_3 = A_1^\dagger A_1 + A_2^\dagger A_2.$$

   Want $\mu_i = \text{diag}(\boldsymbol{\lambda}_i)$

2. **while** $\|\mu_3 - \text{diag}(\boldsymbol{\lambda}_3)\| > \varepsilon$, **do:**
   a. Choose B upper triangular such that $B^\dagger \mu_3 B = \text{diag}(\boldsymbol{\lambda}_3)$,
      Set $\boxed{A_i \leftarrow A_i B}$.
   b. For $i \in 1, 2$, choose B_i upper triangular s.t. $B_i^\dagger \mu_i B_i = \text{diag}(\boldsymbol{\lambda}_i)$,
      Set $\boxed{A_i \leftarrow B_i^\dagger A_i.}$

3. **output** $A_1^\dagger A_1, A_2^\dagger A_2$.

Complexity of moment polytope membership?

The case $\lambda = 0$ is the null-cone problem from Ankit's talk!

1. Is membership in **P**?
   - For tori ($G = \mathbb{C}^n_\times$) Folklore,[SV17]
   - For Horn polytope, by saturation conjecture[MNS12]
2. Is it in **RP**?
   - We think so in general, but no proof yet!
3. Is it in **NP** or **coNP**?
   - In $\mathbf{NP} \cap \mathbf{coNP}$ for $\mathcal{V} = \mathbb{P}(\mathbb{C}^m)$ [BCMW17]
   - Not known in general!

General algorithms

Convert ε-search to an optimization problem

[CF: MISSING!]

Optimization algorithms

Alternating minimization: poly$(1/\varepsilon)$ time [BFGOWW18]

- Tensor products of easy reps e.g. Horn, k-tensors

$\log \mathrm{cap}_\lambda(v)$ can be cast as a geodesically convex program!

Domain is positive-semidefinite matrices; geodesics through P take the form $\sqrt{P}e^{Ht}\sqrt{P}$

Geodesic gradient descent: poly$(1/\varepsilon)$ time [BFGOWW19]

- Any representation, e.g. $\mathcal{V} = \bigwedge^k \mathbb{C}^n, \mathrm{Sym}^k\mathbb{C}^n$, arbitrary quivers

Geodesic trust-regions: poly$(\log(1/\varepsilon), \log\kappa)$ time [BFGOWW19]

- κ is smallest condition-number of an ε-optimizer for $\mathrm{cap}_\lambda(v)$
- polynomial for some interesting cases, e.g. arbitrary quivers with $\lambda = 0$

Open problems

1. Is moment polytope membership in **NP** $\cap$ **coNP**, or even **RP** or **P**?
2. Membership is in **P** for Horn's problem. But how about $\exp(-\,\mathrm{poly})$-search?
3. If (A_1, A_2) a random pair of matrices, does $\mathrm{cap}_\lambda(A_1, A_2)$ have an ε-minimizer with condition number at most

$$\exp(\mathrm{poly}(\log(1/\varepsilon), \langle\lambda\rangle))?$$

Merci!